

Generating functions for spherical harmonics and spherical monogenics

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Abstract. In this paper, we study generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in \mathbb{R}^m . Here spherical monogenics are polynomial solutions of the Dirac equation in \mathbb{R}^m . In particular, we obtain the recurrence formula which expresses the generating function in dimension m in terms of that in dimension $m-1$. Hence we can find closed formulæ of generating functions in \mathbb{R}^m by induction on the dimension m .

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1. Introduction

It is well-known that classical orthogonal polynomials can be defined by their generating functions. For example, the Gegenbauer polynomials C_k^ν are uniquely determined by the generating function

$$\frac{1}{(1 - 2xh + h^2)^\nu} = \sum_{k=0}^{\infty} C_k^\nu(x) h^k \quad (1)$$

where $\nu > 0$, $|x| \leq 1$ and $|h| < 1$ (see e.g. [14, p. 18] or [24, p.173]). In [24], a general framework is developed for a study of properties of polynomial sequences, including the Appell property and generating functions. In this paper, we deal with generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in \mathbb{R}^m .

Orthogonal bases of spherical harmonics are well-known and have been studied for a long time. Spherical harmonics are useful in many theoretical areas and on applications such as structural mechanics, etc. In Clifford analysis, a similar role is played by spherical monogenics. Monogenic functions are defined as Clifford algebra valued solutions f of the equation $\partial f = 0$ where ∂

is the Dirac operator on \mathbb{R}^m . Spherical monogenics are polynomial solutions of the Dirac equation. Since the Dirac operator ∂ factorizes the Laplace operator Δ in the sense that $\Delta = -\partial^2$ Clifford analysis can be understood as a refinement of harmonic analysis. On the other hand, monogenic functions are at the same time a higher dimensional analogue of holomorphic functions of one complex variable. See [3, 13, 17, 16] for an account of Clifford analysis.

The first construction of orthogonal bases of spherical monogenics valid for any dimension was given by F. Sommen, see [25, 13]. In dimension 3, explicit constructions using the standard bases of spherical harmonics were done also by K. Gürlebeck, H. Malonek, I. Cação and S. Bock (see e.g. [1, 6, 7, 8, 9, 10, 11]). From the point of view of representation theory, the standard bases of spherical harmonics are nothing else than examples of the so-called Gelfand-Tsetlin bases, see [23]. V. Souček proposed studying these bases in Clifford analysis. In particular, in [2], it is observed that the complete orthogonal system in \mathbb{R}^3 of [1] and F. Sommen's bases [25, 13] can be both considered as Gelfand-Tsetlin bases. Actually, it turns out that Gelfand-Tsetlin bases in all cases so far studied in Clifford analysis are, by construction, uniquely determined and orthogonal and, in addition, they possess the so-called Appell property, see [22] for a recent survey, [19, 20] for the classical Clifford analysis, [12, 21] for Hodge-de Rham systems and [4, 5] for Hermitian Clifford analysis. Therefore we call them the standard orthogonal bases in the sequel. For a detailed historical account of this topic, we refer to [2].

In this paper, we study generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in \mathbb{R}^m . We obtain the recurrence formula which expresses the generating function in dimension m in terms of that in dimension $m-1$, see below Theorem 1 for spherical harmonics and Theorem 2 for spherical monogenics. Using the recurrence formula, we can obtain closed formulæ of generating functions in \mathbb{R}^m by induction on the dimension m . This is based on the generating function (1) for the Gegenbauer polynomials. It seems that analogous results can be obtained also for Hodge-de Rham systems [21] and even in Hermitian Clifford analysis [5]. But, in the hermitian case, the generating function for the Jacobi polynomials should be used instead of (1).

2. Spherical harmonics

In this section, we study generating functions for spherical harmonics. Let us recall the standard construction of an orthogonal basis in the complex Hilbert space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \text{Ker } \Delta$ of L^2 -integrable harmonic functions $g : \mathbb{B}_m \rightarrow \mathbb{C}$. Here \mathbb{B}_m is the unit ball in \mathbb{R}^m . One proceeds by induction. Of course, the polynomials

$$\text{harm}_{k_2}^{\pm}(x_1, x_2) = (x_1 \pm ix_2)^{k_2} / (k_2!), \quad k_2 \in \mathbb{N}_0 \quad (2)$$

form an orthogonal basis of the space $L^2(\mathbb{B}_2, \mathbb{C}) \cap \text{Ker } \Delta$. To construct the bases in higher dimensions, we need the embedding factors $F_{m,j}^{(k_m)} = F_{m,j}^{(k_m)}(x)$ defined as

$$F_{m,j}^{(k_m)} = |x|_m^{k_m} C_{k_m}^{m/2+j-1}(x_m/|x|_m), \quad x \in \mathbb{R}^m \quad (3)$$

where $x = (x_1, \dots, x_m)$ and $|x|_m = \sqrt{x_1^2 + \dots + x_m^2}$. Then, it is well-known that an orthogonal basis of the space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \text{Ker } \Delta$ is formed by the polynomials

$$\text{harm}_k^\pm(x) = \text{harm}_{k_2}^\pm(x_1, x_2) \prod_{r=3}^m F_{r, k_r^*}^{(k_r)} \quad (4)$$

where $k = (k_2, \dots, k_m) \in \mathbb{N}_0^{m-1}$ and $k_r^* = k_2 + \dots + k_r$. See e.g. [14, p. 35] or [20]. In difference to [20], we use another normalization of the embedding factors $F_{m,j}^{(k_m)}$ and we also change the notation for indices which in turns provides a more elegant expression for generating functions.

Definition 1. We define the generating function H_m^\pm of the orthogonal basis harm_k^\pm , $k \in \mathbb{N}_0^{m-1}$ of spherical harmonics in \mathbb{R}^m by

$$H_m^\pm(x, h) = \sum_{k \in \mathbb{N}_0^{m-1}} \text{harm}_k^\pm(x) h^k$$

whenever the series on the right-hand side converges absolutely. Here $x \in \mathbb{R}^m$, $h = (h_2, \dots, h_m) \in \mathbb{R}^{m-1}$ and $h^k = h_2^{k_2} \dots h_m^{k_m}$.

Obviously, the following result follows easily from (1).

Lemma 1. *We have that*

$$\sum_{k_m=0}^{\infty} F_{m,j}^{(k_m)}(x) h_m^{k_m} = \frac{1}{(1 - 2x_m h_m + h_m^2 |x|_m^2)^{\frac{m}{2}-1+j}}$$

where $|x|_m \leq 1$, $|h_m| < 1$ and $j \in \mathbb{N}_0$.

Now we prove basic properties of the generating functions H_m^\pm .

Theorem 1. *For each $m \geq 2$ there is a neighborhood U_m of 0 in \mathbb{R}^{m-1} such that the following statements hold true.*

- (i) *The generating functions $H_m^\pm(x, h)$ are defined if $|x|_m \leq 1$ and $h \in U_m$.*
- (ii) *For each $k \in \mathbb{N}_0^{m-1}$, we have that*

$$\text{harm}_k^\pm(x) = \frac{1}{k!} \partial^k H_m^\pm(x, h)|_{h=0}, \quad |x|_m \leq 1$$

where $k! = (k_2!) \dots (k_m!)$ and $\partial^k = \partial_{h_2}^{k_2} \dots \partial_{h_m}^{k_m}$.

- (iii) *For $m \geq 3$, $|x|_m \leq 1$ and $h \in U_m$, we have that*

$$H_m^\pm(x, h) = d_m^{1-\frac{m}{2}} H_{m-1}^\pm(\underline{x}, \underline{h}/d_m)$$

where $d_m = 1 - 2x_m h_m + h_m^2 |x|_m^2$, $\underline{x} = (x_1, \dots, x_{m-1})$ and $\underline{h}/d_m = (h_2/d_m, \dots, h_{m-1}/d_m)$.

Proof. We prove this theorem by induction on the dimension m . It is easily seen that the theorem is true for $m = 2$. Indeed, we have that

$$H_2^\pm(x_1, x_2, h_2) = \sum_{k_2=0}^{\infty} \frac{(x_1 \pm ix_2)^{k_2}}{k_2!} h_2^{k_2} = \exp((x_1 \pm ix_2)h_2).$$

Now assume that the theorem is true for $m - 1$. Let $H_{m-1}^\pm(\underline{x}, \underline{h})$ be defined for $\underline{h} \in U_{m-1} = (-\delta_2, \delta_2) \times \cdots \times (-\delta_{m-1}, \delta_{m-1})$ and $|\underline{x}|_{m-1} \leq 1$ and let $|x|_m \leq 1$. It is easy to see that

$$H_m^\pm(x, h) = \sum_{\underline{k}} \left(\sum_{k_m=0}^{\infty} F_{m, k_m^*}^{(k_m)}(x) h_m^{k_m} \right) \text{harm}_{\underline{k}}^\pm(\underline{x}) \underline{h}^{\underline{k}} \quad (5)$$

where the first sum is taken over all $\underline{k} = (k_2, \dots, k_{m-1}) \in \mathbb{N}_0^{m-2}$. By Lemma 1, we have that

$$\sum_{k_m=0}^{\infty} F_{m, k_m^*}^{(k_m)}(x) h_m^{k_m} = d_m^{1-\frac{m}{2}-(k_2+\dots+k_{m-1})}$$

if $|h_m| < 1$. Using this formula and (5), we have that

$$H_m^\pm(x, h) = d_m^{1-\frac{m}{2}} \sum_{\underline{k}} \text{harm}_{\underline{k}}^\pm(\underline{x}) (\underline{h}/d_m)^{\underline{k}} = d_m^{1-\frac{m}{2}} H_{m-1}^\pm(\underline{x}, \underline{h}/d_m)$$

whenever $h \in U_m = (-\delta_2/4, \delta_2/4) \times \cdots \times (-\delta_{m-1}/4, \delta_{m-1}/4) \times (-1/2, 1/2)$. Indeed, $d_m \geq (1 - h_m|x|_m)^2 > 1/4$ if $|h_m| < 1/2$. Hence, if $h \in U_m$ we have that $\underline{h}/d_m \in U_{m-1}$ and, by (5), we can easily see that some rearrangement of the power series defining $H_m^\pm(x, h)$ converges at h . Then Abel's Lemma [18, Proposition 1.5.5, p. 23] proves that this power series converges absolutely on the whole U_m , which finishes the proof of the theorem. \square

Using the recurrence formula (iii) of Theorem 1, we can find closed formulæ of generating functions for spherical harmonics in \mathbb{R}^m by induction on the dimension m .

Corollary 1. *In particular, we have the following formula*

$$H_3^\pm(x_1, x_2, x_3, h_2, h_3) = \frac{1}{(1 - 2x_3h_3 + h_3^2|x_3|^2)^{1/2}} \exp\left(\frac{(x_1 \pm ix_2)h_2}{1 - 2x_3h_3 + h_3^2|x_3|^2}\right).$$

Remark 1. It is well-known that an orthogonal basis of real valued spherical harmonics in \mathbb{R}^m is formed by the polynomials $\Re \text{harm}_k^+$, $\Im \text{harm}_k^+$, $k \in \mathbb{N}_0^{m-1}$. Here $\Re z$ and $\Im z$ are the real and imaginary part of the complex number z . Hence the corresponding generating functions are $\Re H_m^+$, $\Im H_m^+$.

Remark 2. If one replaces in the definition of the orthogonal basis (4) the polynomials $\text{harm}_{k_2}^\pm(x_1, x_2) = (x_1 \pm ix_2)^{k_2}/(k_2!)$ with

$$\overline{\text{harm}_{k_2}^\pm}(x_1, x_2) = (x_1 \pm ix_2)^{k_2}, \quad (6)$$

the corresponding generating functions \overline{H}_m^\pm are definitely different from H_m^\pm but they obviously satisfy again Theorem 1. In particular, we have that

$$\overline{H}_2^\pm(x_1, x_2, h_2) = \sum_{k_2=0}^{\infty} (x_1 \pm ix_2)^{k_2} h_2^{k_2} = \frac{1 - (x_1 \mp x_2 i) h_2}{1 - 2x_1 h_2 + h_2^2 |x|_2^2}.$$

3. Spherical monogenics

In this section, we introduce and investigate generating functions for spherical monogenics. For an account of Clifford analysis, we refer to [3, 13, 17, 16]. Denote by $\mathcal{C}\ell_m$ either the real Clifford algebra $\mathbb{R}_{0,m}$ or the complex one \mathbb{C}_m , generated by the vectors e_1, \dots, e_m such that $e_j^2 = -1$ for $j = 1, \dots, m$. As usual, we identify a vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ with the element $x_1 e_1 + \dots + x_m e_m$ of the Clifford algebra $\mathcal{C}\ell_m$. Let $G \subset \mathbb{R}^m$ be open. Then a continuously differentiable function $f : G \rightarrow \mathcal{C}\ell_m$ is called monogenic if it satisfies the equation $\partial f = 0$ on G where the Dirac operator ∂ is defined as

$$\partial = e_1 \partial_{x_1} + \dots + e_m \partial_{x_m}. \quad (7)$$

Denote by $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \text{Ker } \partial$ the space of L^2 -integrable monogenic functions $g : \mathbb{B}_m \rightarrow \mathcal{C}\ell_m$. It is well-known that $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \text{Ker } \partial$ forms the right $\mathcal{C}\ell_m$ -linear Hilbert space. Let us recall a construction of an orthogonal basis in this space, see [20] for more details. It is easy to see that the polynomials

$$\text{mon}_{k_2}(x_1, x_2) = (x_1 - e_{12} x_2)^{k_2} / (k_2!), \quad k_2 \in \mathbb{N}_0 \quad (8)$$

form an orthogonal basis of the space $L^2(\mathbb{B}_2, \mathcal{C}\ell_2) \cap \text{Ker } \partial$. Here we write $e_{12} = e_1 e_2$ as usual. To construct the bases in higher dimensions, we need the embedding factors $X_{m,j}^{(k_m)} = X_{m,j}^{(k_m)}(x)$ defined as

$$X_{m,j}^{(k_m)} = \frac{m-2+k_m+2j}{m-2+2j} F_{m,j}^{(k_m)}(x) + F_{m,j+1}^{(k_m-1)}(x) \underline{x} e_m, \quad x \in \mathbb{R}^m \quad (9)$$

where $\underline{x} = x_1 e_1 + \dots + x_{m-1} e_{m-1}$, $F_{m,j}^{(k_m)}$ are given in (3) and $F_{m,j+1}^{(-1)} = 0$. Then it is well-known that an orthogonal basis of the space $L^2(\mathbb{B}_m, \mathcal{C}\ell_m) \cap \text{Ker } \partial$ is formed by the polynomials

$$\text{mon}_k(x) = X_{m,k_m-1}^{(k_m)} X_{m-1,k_m-2}^{(k_m-1)} \dots X_{3,k_2}^{(k_3)} \text{mon}_{k_2}(x_1, x_2) \quad (10)$$

where $k = (k_2, \dots, k_m) \in \mathbb{N}_0^{m-1}$ and $k_r^* = k_2 + \dots + k_r$. Let us remark that due to non-commutativity of the Clifford multiplication the order of factors in the product (10) is important. See [20] for more details. In comparison with [20], we use another normalization of the embedding factors $X_{m,j}^{(k_m)}$ and we also change the notation for indices to get a nice expression for generating functions.

Definition 2. We define the generating function M_m of the orthogonal basis mon_k , $k \in \mathbb{N}_0^{m-1}$ of spherical monogenics in \mathbb{R}^m by

$$M_m(x, h) = \sum_{k \in \mathbb{N}_0^{m-1}} mon_k(x) h^k$$

whenever the series on the right-hand side converges absolutely. Here $x \in \mathbb{R}^m$ and $h = (h_2, \dots, h_m) \in \mathbb{R}^{m-1}$.

In particular, it is easily seen that

$$M_2(x_1, x_2, h_2) = \sum_{k_2=0}^{\infty} \frac{(x_1 - e_{12}x_2)^{k_2}}{k_2!} h_2^{k_2} = \exp((x_1 - e_{12}x_2)h_2).$$

Here $\exp((x_1 - e_{12}x_2)h_2) = \exp(x_1h_2)(\cos(x_2h_2) - e_{12}\sin(x_2h_2))$. To study the generating functions in higher dimensions we need to know the generating function of the embedding factors $X_{m,j}^{(k_m)}$.

Lemma 2. *We have that*

$$\sum_{k_m=0}^{\infty} X_{m,j}^{(k_m)}(x) h_m^{k_m} = \frac{1 + xh_me_m}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{m/2+j}}$$

where $|x|_m \leq 1$, $|h_m| < 1$ and $j \in \mathbb{N}_0$.

Proof. Put $\nu = m/2 - 1 + j$. By (9), the series we want to sum up is equal to

$$\sum_{k_m=0}^{\infty} \frac{k_m + 2\nu}{2\nu} F_{m,j}^{(k_m)}(x) h_m^{k_m} + \sum_{k_m=1}^{\infty} F_{m,j+1}^{(k_m-1)}(x) h_m^{k_m} \underline{x}e_m = \Sigma_1 + \Sigma_2.$$

Obviously, by Lemma 1, we get that

$$\Sigma_2 = \frac{\underline{x}h_me_m}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{\nu+1}}.$$

Moreover, using Lemma 1 again, we have that

$$\Sigma_1 = \frac{h_m}{2\nu} \sum_{k_m=1}^{\infty} F_{m,j}^{(k_m)}(x) k_m h_m^{k_m-1} + \frac{1}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{\nu}}$$

and hence

$$\Sigma_1 = \frac{h_m}{2\nu} \frac{d}{dh_m} \frac{1}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{\nu}} + \frac{1}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{\nu}},$$

which gives

$$\Sigma_1 = \frac{1 - x_mh_m}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{\nu+1}}.$$

Finally, we conclude that

$$\Sigma_1 + \Sigma_2 = \frac{1 + xh_me_m}{(1 - 2x_mh_m + h_m^2|x|_m^2)^{m/2+j}},$$

which finishes the proof. \square

Now we can prove basic properties of the generating functions M_m quite similarly as in the harmonic case if, in this case, we use Lemma 2 instead of Lemma 1. Then we obtain the following result.

Theorem 2. *For each $m \geq 2$ there is a neighborhood U_m of 0 in \mathbb{R}^{m-1} such that the following statements hold true.*

- (i) *The generating functions $M_m(x, h)$ are defined if $|x|_m \leq 1$ and $h \in U_m$.*
- (ii) *For each $k \in \mathbb{N}_0^{m-1}$, we have that*

$$\text{mon}_k(x) = \frac{1}{k!} \partial^k M_m(x, h)|_{h=0}, \quad |x|_m \leq 1$$

where $k! = (k_2!) \cdots (k_m!)$ and $\partial^k = \partial_{h_2}^{k_2} \cdots \partial_{h_m}^{k_m}$.

- (iii) *For $m \geq 3$, $|x|_m \leq 1$ and $h \in U_m$, we have that*

$$M_m(x, h) = (1 + x h_m e_m) d_m^{-\frac{m}{2}} M_{m-1}(\underline{x}, \underline{h}/d_m)$$

where $d_m = 1 - 2x_m h_m + h_m^2 |x|_m^2$, $\underline{x} = (x_1, \dots, x_{m-1})$ and $\underline{h}/d_m = (h_2/d_m, \dots, h_{m-1}/d_m)$.

Using the recurrence formula (iii) of Theorem 2, we can find closed formulæ of generating functions for spherical monogenics in \mathbb{R}^m by induction on the dimension m .

Corollary 2. *In particular, we have the following formula*

$$M_3(x_1, x_2, x_3, h_2, h_3) = \frac{1 + x h_3 e_3}{(1 - 2x_3 h_3 + h_3^2 |x|_3^2)^{3/2}} \exp \left(\frac{(x_1 - e_{12} x_2) h_2}{1 - 2x_3 h_3 + h_3^2 |x|_3^2} \right).$$

Remark 3. If one replaces in the definition of the orthogonal basis (10) the polynomials $\text{mon}_{k_2}(x_1, x_2) = (x_1 - e_{12} x_2)^{k_2} / (k_2!)$ with

$$\overline{\text{mon}}_{k_2}(x_1, x_2) = (x_1 - e_{12} x_2)^{k_2}, \quad (11)$$

the corresponding generating functions \overline{M}_m are different from M_m but they obviously satisfy again Theorem 2. In particular, we have that

$$\overline{M}_2(x_1, x_2, h_2) = \sum_{k_2=0}^{\infty} (x_1 - e_{12} x_2)^{k_2} h_2^{k_2} = \frac{1 - (x_1 + e_{12} x_2) h_2}{1 - 2x_1 h_2 + h_2^2 |x|_2^2}.$$

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